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ON THE MELTING OF SOLIDS NEAR A HOT MOVING INTERFACE, WITH PARTICULAR REFERENCE TO BEDS OF GRANULAR POLYMERS

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Abstract—Some similarity solutions are presented for velocity and temperature profiles within the thin film of fluid that forms between a moving hot interface and a melting solid. The use of dimensionless variables brings out the role of various dimensionless groups, in particular when a shear and temperature-dependent viscous fluid is considered.

A simplified heuristic method of solution of the full problem for the case of plasticating extruders is discussed.

NOMENCLATURE

A, dimensionless constant;
b, temperature coefficient of viscosity;
B, temperature ratio based on b and plate temperature;
B^*, modified temperature ratio;
C_s, specific heat of solid;
C_f, specific heat of fluid;
C, constant;
C_o, viscosity constant;
Gn, Brinkmann number;
H, depth of solid bed;
k, thermal conductivity;
L, characteristic length of apparatus;
M, inverse Graetz number; temperature ratio based on latent heat of fusion and plate temperature;
p, pressure;
Q, volume flux in z-direction;
S, dimensionless variable in s-direction;
v, y component of fluid velocity;
w, z component of fluid velocity;
W_o, velocity of moving plate;
y, z, Cartesian coordinates.

Greek symbols

α, dimensionless depth of layer;
β, power-law coefficient of viscosity;
δ, layer thickness;
$\bar{\delta}$, representative layer thickness;
δ_o, constant layer thickness;
κ, thermal diffusivity based on k, ρ_f, C_f;
Λ^*, apparent latent heat of fusion;
Λ, latent heat of fusion;
$\hat{\Theta}$, dimensionless mean temperature;
Θ, dimensionless temperature;
θ, temperature;
θ_s, bulk solid temperature;
θ_0, temperature of plate at $y = 0$;
μ, fluid viscosity;
$\bar{\mu}$, representative fluid viscosity;
λ, constant;
ρ_f, fluid density;
ρ_s, solid density;
η, dimensionless coordinate in y-direction;
ζ, dimensionless coordinate in z-direction;
Ψ, dimensionless velocity in z-direction;
Φ, dimensionless velocity in y-direction;
$\phi, \psi, \bar{\psi}$, similarity functions for velocity;
$\theta, \bar{\theta}$, similarity functions for temperature;
χ, dimensionless flux.

1. INTRODUCTION

The physical problem

WE CONSIDER the two-dimensional situation shown diagrammatically in Fig. 1. A hot plate $y = 0$ held at constant temperature $\theta = \theta_0 > 0$ moves with constant velocity W_o in the z-direction. A deformable but coherent bed of solid whose melting point is $\theta = 0$ is pushed against the hot plate in the region $z > 0$. A barrier occupies the line $z = 0$, $y > 0$; in practice we may assume that some molten material leaks to the point $(0, 0)$ and so there is some initial flux of molten material at $z = 0$, but we do not wish to be too precise about this aspect at this stage.

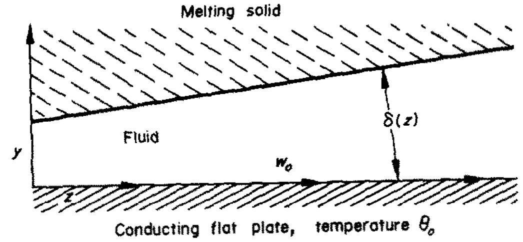

FIG. 1. Diagram of flow region.

The steady-state physical situation is known from experience to involve a thin layer of highly sheared liquid in the neighbourhood of the hot plate across which heat is conducted to melt the solid; if the viscosity of the melt is high enough, significant heat may be generated within the fluid layer which adds to the rate of melting. Clearly the depth of the fluid layer can be expected to increase with z. We seek here to calculate the rate of melting, particularly with reference to the situation that arises in the melting of thermoplastic powders or granules, which have low thermal conductivity.

The full mechanics of the situation is necessarily complex, in that the solid cannot be taken to be wholly rigid if a steady-state situation is to ensue (the rate of melting at various stations z will not in general be equal and so the solid must deform if a melting interface fixed in space is to be achieved). On the other hand, the solid material is usually sufficiently well packed for it to resist in large measure the local shear stresses at its melting interface caused by the drag of the fluid above it. Thus, in many cases it seems that one can suppose the interface material to have no velocity in the z-direction and a very small one in the negative y-direction.

We shall develop the partial differential equations and associated boundary conditions that govern flow in the thin liquid layer making full use of obvious and well-known approximations. These, when made dimensionless, yield the relevant dimensionless parameters that determine the possible regimes of melting. The problem (of solving these equations) only becomes determinate if the thickness of the liquid layer is not prescribed, when some additional assumption is made about the pressure distribution in the liquid layer; this latter has to be related to the stress-deformation properties of the solid bed. It will be argued that a suitable approximation is provided by supposing the flow to be drag-flow, i.e. that the pressure is effectively constant everywhere.

2. MATHEMATICAL FORMULATION

If v and w are the velocities in the y and z directions respectively, and the fluid is taken to be incompressible (with density ρ_f) the continuity equation is simply

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{1}$$

If we take $\delta(z)$ to be the thickness of the layer, then the statement that δ is small is to be interpreted as $\mathrm{d}\delta/\mathrm{d}z \ll 1$; the order of this term is $\bar{\delta}/L$ where L is some characteristic length of the apparatus, such as the depth or width of the bed and $\bar{\delta}$ is a representative layer thickness. We can later confirm that these quantities are indeed small. On the assumption that $\mathrm{d}\delta/\mathrm{d}z$ is very much less than unity, then we may at once invoke the lubrication approximation [1] as regards the momentum equations, particularly since we can assume in cases of interest to us that the Reynolds number $\rho_f W_o \delta/\bar{\mu}$ is small (where $\bar{\mu}$ is a representative fluid viscosity). This yields the stress equilibrium relation

$$\frac{\partial p}{\partial z}=\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{2}$$

where p is pressure and μ may be a function of temperature and shear rate $\partial w/\partial y$.

The energy equation will here be chosen to include convection, conduction and generation terms. This yields, for the steady state, bearing in mind that $\partial^2/\partial z^2 \ll \partial^2/\partial y^2$ in the full conduction term, and assuming that specific heat C_f and thermal conductivity k are constant,

$$\rho_f C_f\left(w\frac{\partial\theta}{\partial z}+v\frac{\partial\theta}{\partial y}\right)=k\frac{\partial^2\theta}{\partial y^2}+\mu\left(\frac{\partial w}{\partial y}\right)^2. \tag{3}$$

In common with earlier work we shall write

$$\mu=C_o\mathrm{e}^{-b\theta}\left[\left(\frac{\partial w}{\partial y}\right)^2\right]^{-\beta} \tag{4}$$

giving an exponential temperature dependence and a power-law shear-rate dependence.

Five boundary conditions are seen at once to be

$$w=W_o,\quad v=0,\quad \theta=\theta_0\quad \text{on}\quad y=0. \tag{5}$$

$$w=0,\quad \theta=0\quad \text{on}\quad y=\delta. \tag{6}$$

If melting is to take place, and the position of the interface is to be stationary, then $v(\delta)$ will be determined by the rate at which heat flows into the solid bed from the liquid. Thus the last boundary condition becomes

$$k\frac{\partial\theta}{\partial y}=\rho_f\Lambda v\quad \text{at}\quad y=\delta \tag{7}$$

where Λ is the effective latent heat of fusion of the fluid. (This would in general be the true latent heat plus the amount of heat $-C_s\theta_s$ required to raise the cold solid from its bulk temperature $-\theta_s$ to its melting temperature.) It will be noticed that the equations (1)–(7) involve two unknown functions $p(z)$ and $\delta(z)$. Without further information, they are not determinate. If $p(z)$ or $\delta(z)$ is assumed known they at once become so.

(a) *The uniform gap approximation*

If we suppose δ to be a constant δ_o, then it can be shown by order of magnitude arguments that the pressure differences arising within the liquid layer along its full length will be of order L/δ_o greater than the shear stresses at the interface. Thus in simple terms the normal stresses at the interface will dominate the shear stresses.

The simplest way to derive the result is to consider the case of a Newtonian fluid of constant viscosity melting uniformly along its length L across a liquid film of thickness δ_o where the lower surface is dragging the fluid at a velocity W_o (i.e. precisely the circumstances we have outlined above). The flux Q in the z direction is given by

$$Q=\tfrac{1}{2}W_o\delta_o-\tfrac{1}{12}\frac{\delta_o^3}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}.$$

The second term must be at least of the same order as the first if Q is to be zero at $z=0$. The shear stress will be of order $\mu Q/\delta_o^2$; the pressure drop between $z=0$ and $z=L$ will be of order $\mu QL/\delta_o^3$; their ratio is therefore L/δ_o as required.

The next stage in the argument is to decide whether these large normal pressures can indeed be sustained by the solid bed. In practice, the bed will be granular and so although non-isotropic stress can be sustained, all three principal stresses will always be of the same magnitude to within a factor of about 3. Thus the pressure $\mu QL/\delta_o^3$ must not be large compared with other pressures generated. When the case of a plasticating screw extruder is considered, it is found that the pressures generated by the melt pumping mechanisms are too small to balance those implied by $\delta_o=$ constant. However, to carry out the order-of-magnitude calculation, an estimate must be made of δ_o; this is done by noting that the solid bed of depth H melts over a length of order $20L$ and so we deduce that $\delta_o \simeq H/10$. The melt pumping mechanism over the same length leads to pressure differences of order $20\mu W_o L/H^2 \simeq 20\mu QL/H^2\delta_o$; the relevant ratio turns out thus to be 100:1. We can therefore deduce that a uniform gap does not arise.†

(b) *The constant pressure approximation*

From the conclusion of the last sub-section we can assume that normal stresses generated on the thin liquid layer above the melting interface will be sufficient to deform the solid bed; the constant depth approximation is not relevant. Indeed, we may argue that, since the interfacial normal stresses that the bed can sustain will be no higher than the interfacial shear stresses, a reasonable approximation to the flow in the liquid layer will be given by $p=$ constant, $\mathrm{d}p/\mathrm{d}z=0$. What we are really asserting is that those variations in $p(z)$ that do arise will only have a small effect on the liquid layer flow, which can be taken to be drag-flow everywhere.

Making this assumption, we are now in a position to make equations (1)–(7) dimensionless. Anticipating the methods of solution we shall subsequently use, and remembering that $\delta_o/L \ll 1$, we choose as dimensionless coordinates

$$\zeta = z/L, \quad \eta = y/\delta \tag{8}$$

where

$$\alpha = \delta/\delta_o \tag{9}$$

(a function of ζ only) is, by definition, of order one, as are its derivatives. We also choose, for obvious reasons, a dimensionless temperature

$$\Theta = \theta/\theta_0 \tag{10}$$

and dimensionless velocities

$$\Psi = w/W_o; \quad \Phi = vL/W_o\delta_o. \tag{11}$$

The boundary conditions (5) and (6) become

$$\Psi(0,\zeta)=1; \quad \Phi(0,\zeta)=0; \quad \Theta(0,\zeta)=1 \tag{12}$$

$$\Psi(1,\zeta)=0; \quad \Theta(1,\zeta)=0 \tag{13}$$

†The argument would be more complicated if we took account of heat generation and convection but the conclusion would be essentially the same; *a posteriori* use of a mean δ based on the constant pressure approximation effectively confirms the deduction.

thus confirming the choice of scale temperature and velocity for Θ and Ψ. The melting condition (7) becomes

$$\frac{k\theta_0 L}{\rho_f \Lambda W_o \delta_o^2}\frac{\partial\Theta}{\partial\eta} = \alpha\Phi \quad \text{at} \quad \eta=1. \tag{14}$$

This is the boundary condition that determines the physical process and so the dimensionless parameter $k\theta_0 L/\rho_f \Lambda W_o \delta_o^2$ should be $O(1)$. This then is the obvious way of choosing δ_o to give the correct scaling for the various variables. We may therefore write

$$\delta_o = \left(\frac{k\theta_0 L}{\rho_f \Lambda W_o}\right)^{\frac{1}{2}}. \tag{15}$$

Equations (1)–(3) then become, using (4),

$$\frac{\partial\Psi}{\partial\zeta} - \frac{\alpha'}{\alpha}\eta\frac{\partial\Psi}{\partial\eta} + \frac{1}{\alpha}\frac{\partial\Phi}{\partial\eta} = 0 \tag{16}$$

$$\frac{\partial}{\partial\eta}\left\{e^{-B\Theta}\left(\frac{\partial\Psi}{\partial\eta}\right)^{1-2\beta}\right\} = 0 \tag{17}$$

and

$$M^{-1}\left(\Psi\frac{\partial\Theta}{\partial\zeta} - \frac{\alpha'}{\alpha}\Psi\eta\frac{\partial\Theta}{\partial\eta} + \frac{1}{\alpha}\Phi\frac{\partial\Theta}{\partial\eta}\right) = \frac{1}{\alpha^2}\frac{\partial^2\Theta}{\partial\eta^2} + \frac{Gn}{\alpha^{2-2\beta}}\left(\frac{\partial\Psi}{\partial\eta}\right)^{2-2\beta} e^{-B\Theta} \tag{18}$$

where

$$B = b\theta_0, \quad M = \Lambda/C_f\theta_0, \quad Gn = \frac{C_o W_o^{2-3\beta} L^\beta}{(k\theta_0)^{1-\beta}\Lambda^\beta}. \tag{19}$$

B, M and Gn can all be thought of as ratios of temperatures. B measures the ratio of the imposed temperature difference to the natural rheological temperature scale b^{-1}; if B is small, then the flow is rheologically almost isothermal, and there is no Θ effect in equations (16) and (17). M measures the ratio of the change of internal energy involved in melting to the change involved in raising the temperature of the liquid by θ_0; if M is large, melting will be slow and convection of heat measured by the LHS of equation (18) will be small compared with conduction, measured by the first term on the RHS. Gn is a Brinkmann number and measures the ratio of the equilibrium rise in temperature in the liquid layer due to viscous heating to the imposed temperature difference. If Gn is small, generation is unimportant. If it is large, then the argument used in choosing δ_o as in (15) is unsatisfactory in that generation can make $\Theta(\eta)$ large compared with unity; θ_0 is no longer a suitable scale temperature. Similarly if M is small.

(c) *Similarity solutions*

The set of equations (16)–(18) together with the boundary conditions (12), (13) and

$$\frac{\partial\Theta}{\partial\eta} = \alpha\Phi \quad \text{at} \quad \eta=1 \tag{20}$$

(obtained from (14) and (15)) can readily be seen to admit the similarity solution

$$\Psi = \psi(\eta); \quad \Phi = \phi(\eta)\alpha'(\zeta); \quad \Theta = \theta(\eta) \tag{21}$$

provided $\beta = 0$.

The variables ψ, ϕ, θ and α are now given by

$$\eta\psi' - \phi' = 0 \tag{22}$$

$$(e^{-B\theta}\psi')' = 0 \tag{23}$$

$$M^{-1}A(\eta\psi-\phi)\theta' + \theta'' + Gn(\psi')^2 e^{-B\theta} = 0 \tag{24}$$

$$\alpha\alpha' = A \tag{25}$$

with boundary conditions

$$\psi(0) = 1,\ \phi(0) = 0,\ \theta(0) = 1; \tag{26}$$

$$\psi(1) = 0,\ \theta(1) = 0,\ \theta'(1) = A\phi(1). \tag{27}$$

It will be noticed that the constant A is in a sense a characteristic (or eigen) value of the solution, in that (26) and (27) provide six boundary conditions to the fifth-order set of equations (22)–(24). It is determined by any given choice of B, M and Gn. (25) is solved to give $\alpha = (2A\zeta)^{\frac{1}{2}}$.

Equation (22) can be integrated to give

$$\phi(1) = \int_0^1 \eta\psi'\,d\eta = -\int_0^1 \psi\,d\eta$$

and so

$$A = -\theta'(1)\bigg/\int_0^1 \psi\,d\eta.$$

Although the similarity solution will not hold strictly when $\beta > 0$, a locally valid approximate solution could be obtained by taking

$$\mu_o = C_o\left(\frac{W_o}{\delta}\right)^{-2\beta}$$

3. SOLUTION METHODS

It is not immediately clear how the full set of equations (16)–(18) and boundary conditions (12), (13) and (20) could be solved by direct numerical means. Since (16) and (18) are partial differential equations involving first order derivatives in ζ, we may assume that initial conditions at $\zeta = 0$ would be given. We note that $\Psi(\eta, 0)$ would have to be everywhere positive and that $\Psi(\eta, 0)$ and $\Theta(\eta, 0)$ would be specified subject to equation (17) and the relevant boundary conditions on Ψ and Θ in (12) and (13). Φ can be obtained formally by integration from (16); $\alpha(0)$ would also be specified. A coupled pair of integro-differential equations for $\Psi(\eta, \zeta)$ and $\alpha(\zeta)$ result. Iterative schemes can be devised, but these prove to be expensive in computing time and have furthermore to be carefully categorized in terms of B, M, Gn and the initial conditions. In practice what one seeks to calculate is the flux

$$\chi(\zeta) = \frac{Q(z)}{W_o\delta} = \int_0^1 \psi(\eta, \zeta)\,d\eta \tag{28}$$

and the associated mean temperature

$$\hat{\Theta}(\zeta) = \int_0^1 \Psi(\eta, \zeta)\Theta(\eta, \zeta)\,d\eta/\chi(\zeta) \tag{29}$$

together with $\alpha(\zeta)$.

We shall now try to develop a simple approximate method for deriving these, based on a few special solutions. Further details of these are given in [2], copies of which can be obtained on demand.

It will be useful in the case of similarity solutions, using (28), (15) and our earlier results for A and α, to write

$$Q = W_o\delta\chi = \left(\frac{-2k\theta_0 W_o z\chi\theta'(1)}{\rho_f\Lambda}\right)^{\frac{1}{2}} = \left(\frac{-2\kappa W_o\chi\theta'(1)z}{M}\right)^{\frac{1}{2}}.$$

It is also worth noting that by direct integration of (22) and (23) we have

$$\phi = \int_0^\eta \xi\psi'\,d\xi;\quad \psi' e^{-B\theta} = \psi'_1.$$

Thus (24) can be written

$$\theta'' + \frac{A}{M}\int_0^\eta \psi\,d\xi\,\theta' + Gn\psi'_1\psi' = 0.$$

One further integration yields when $\eta = 1$

$$\frac{A}{M}\chi\hat{\theta} = \theta'_1 - \theta'_0 - Gn\psi'_1. \tag{29a}$$

This is the obvious dimensionless form of the integrated energy balance.

(a) *The case* M $\gg 1$, B $\ll 1$, $\beta = 0$

This is the full lubrication approximation for near isothermal fluid. The relevant solution can readily be seen to be

$$A = (Gn + 2) \tag{30}$$

$$Q = \left(W_o(1+\tfrac{1}{2}Gn)\frac{k\theta_0}{\rho_f\Lambda}z\right)^{\frac{1}{2}} = \left(W_o\frac{(1+\frac{1}{2}Gn)}{M}\kappa z\right)^{\frac{1}{2}} \tag{31}$$

and

$$\hat{\Theta} = (\tfrac{2}{3} + \tfrac{1}{12}Gn)\theta_0. \tag{32}$$

(b) *The case* M $\gg 1$, Gn $\ll 1$

This retains the full lubrication approximation, but neglects generation.

If we write $B^* = B/(1-2\beta)$, then the relevant solutions are

$$A = \frac{B^*(e^{B^*}-1)}{(e^{B^*}-(1+B^*))} \tag{33}$$

$$\chi = \frac{1}{B^*} + \frac{1}{e^{B^*}-1} = \frac{1}{A};\qquad Q = \left(\frac{W_o 2(e^{B^*}-B^*-1)}{B^*(e^{B^*}-1)}\frac{k\theta_0 z}{\Lambda}\right)^{\frac{1}{2}} \tag{34}$$

and

$$\hat{\theta} = \frac{2e^{B^*}(B^*-1)-(B^{*2}-2)}{2B^*(e^{B^*}-(1+B^*))}. \tag{35}$$

(c) *The case* M $\gg 1$, $\beta = 0$

This extends the results of Section 3(a) to a fluid with temperature-dependent viscosity.

Using results given in [3], we obtain

$$\chi = \tfrac{1}{2} + \frac{e^B-1}{BGn} - \frac{1}{\psi' Gn} \tag{36}$$

where

$$\psi_1' = \frac{\tanh^{-1}\left((1-\psi_2)\left(\frac{BGn}{2\,e^{B\lambda}}\right)^{\frac{1}{2}}\right)+\tanh^{-1}\left(\psi_2\left(\frac{BGn}{2\,e^{B\lambda}}\right)^{\frac{1}{2}}\right)}{(\frac{1}{2}BGn\,e^{B\lambda})^{\frac{1}{2}}} \tag{37}$$

and

$$\psi_2 = \tfrac{1}{2}+\frac{e^B-1}{BGn} \tag{38}$$

$$e^{B\lambda} = \frac{BGn}{8}+\tfrac{1}{2}(e^B+1)+\frac{(e^B-1)^2}{2BGn}. \tag{39}$$

Using

$$\theta'(1) = \left(\frac{2Gn(e^{B\lambda}-1)}{B}\right)^{\frac{1}{2}}\psi_1'$$

we can evaluate Q. $\hat{\theta}$ is not readily expressible in explicit form.

(d) *The case* $M \gg 1$, $\beta = 0$, $Gn \gg 1$

It may readily be seen that ψ_2 and $\chi \sim \frac{1}{2}$, that $\psi_1' \sim 8\ln Gn/BGn$, that $\psi \sim \frac{1}{2}$ almost everywhere and that

$$\hat{\theta} \sim \frac{1}{B}\ln(\tfrac{1}{8}BGn), \quad Q \sim \left(\frac{\kappa W_o 2(\ln Gn)}{MB}\right)^{\frac{1}{2}} \tag{40}$$

provided B is large enough.

From the results in sub-sections 3(a–d) we may readily calculate Q, $\hat{\theta}$, and α for all values of B and Gn, on the assumption that $M \gg 1$.

(e) *The case* $\beta = 0$, $B \ll 1$

The velocities ψ and ϕ become directly $\psi = 1-\eta$; $\phi = -\frac{1}{2}\eta^2$. Equation (22) becomes

$$\theta''+\frac{A}{M}(\eta-\tfrac{1}{2}\eta^2)\theta'+Gn = 0. \tag{41}$$

This may be integrated once to give

$$\theta' = e^{-\frac{A}{2M}(\eta^2-\frac{1}{3}\eta^3)}\left(\theta_0'-Gn\int_0^\eta e^{\frac{A}{2M}(\xi^2-\frac{1}{3}\xi^3)}\,d\xi\right) \tag{42}$$

and again to yield

$$\theta = 1+\int_0^\eta e^{-\frac{A}{2M}(\xi^2-\frac{1}{3}\xi^3)}\left(\theta_0'-Gn\int_0^\eta e^{\frac{A}{2M}(\zeta^2-\frac{1}{3}\zeta^3)}\,d\zeta\right)d\xi \tag{43}$$

where use has been made of $\theta(0) = 1$. The conditions $\theta(1) = 0$ and $\theta'(1) = A\phi(1)$ yield the implicit relations

$$-1 = \int_0^1 e^{-\frac{A}{2M}(\xi^2-\frac{1}{3}\xi^3)}\left(\theta_0'-Gn\int_0^\xi e^{\frac{A}{2M}(\zeta^2-\frac{1}{3}\zeta^3)}\,d\zeta\right)d\xi \tag{44}$$

and

$$-\tfrac{1}{2}A = e^{-\frac{A}{3M}}\left(\theta_0'-Gn\int_0^1 e^{\frac{A}{2M}(\xi^2-\frac{1}{3}\xi^3)}\,d\xi\right) \tag{45}$$

from which θ' may be eliminated and A obtained as an implicit function of M and Gn. Since $\chi = \frac{1}{2}$, we find that

$$Q = \left(\frac{2\kappa W_o Az}{M}\right)^{\frac{1}{2}}. \tag{46}$$

If $Gn \ll 1$, we note that

$$1 = A\,e^{\frac{A}{3M}}\int_0^1 e^{-\frac{A}{M}(\xi^2-\frac{1}{3}\xi^2)}\,d\xi$$

and that

$$\hat{\theta} = M(e^{\frac{A}{3M}}-1). \tag{47}$$

(f) *The case* $\beta = 0$, $M \ll 1$

The equation (24) is dominated by the convective term, and so $\theta' \ll 1$, except in the neighbourhood of $\eta = 0$. We try to rescale the η coordinate using

$$S = (A/M)^{\frac{1}{2}}\eta \tag{48}$$

to give

$$\bar{\theta}_{SS}+\bar{\theta}_S\int_0^S \bar{\psi}(t)\,dt = O(M/A) \tag{49}$$

where $\bar{\theta}(S) = \theta(\eta)$, $\bar{\psi}(S) = \psi(\eta)$.

Unless B is large, the equation

$$\bar{\psi}_S = (M/A)^{\frac{1}{2}}\psi_\eta = \psi_1'\,e^{B\bar{\theta}(S)} \tag{50}$$

means that $\bar{\psi}(S)$ is essentially $1-O(M/A)^{\frac{1}{2}}$ for S of order unity, and so (49) becomes, to order $(M/A)^{\frac{1}{2}}$,

$$\bar{\theta}_{SS}+S\bar{\theta}_S = 0. \tag{51}$$

This integrates to give

$$\bar{\theta}_S = \bar{\theta}_{S0}\,e^{-S^2/2} \tag{52}$$

and again to yield

$$\bar{\theta} \sim 1+\bar{\theta}_{S0}\sqrt{(\pi/2)}\,\mathrm{erf}(S). \tag{53}$$

This, when rewritten in terms of η, can be used in the relation $\theta(1) = 0$ to give $\bar{\theta}_S(0)$ to order $(M/A)^{\frac{1}{2}}$, since $\mathrm{erf}(A/M)^{\frac{1}{2}} \sim 1$. Thus

$$\theta_\eta(0) = -\left(\frac{2}{\pi}\frac{A}{M}\right)^{\frac{1}{2}}+O(1). \tag{54}$$

To the same order of approximation $\chi = \frac{1}{2}$ and $\hat{\theta} = 0$.

What we need, of course, is to discover how A depends on M. Equation (41) gives

$$\theta_1' = -2Gn\frac{M}{A}$$

and so from

$$A = -\theta_1'/\chi$$

we get

$$A \simeq 2(MGn)^{\frac{1}{2}} \tag{55}$$

confirming that $(M/A) \simeq (M/4Gn)^{\frac{1}{2}}$ is small, and making

$$Q \simeq \left(\frac{4\kappa W_o Gn^{\frac{1}{2}}z}{M^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{56}$$

A matched asymptotic expansion to improve the result could be carried out but is not needed here. If Gn is small, then a special analysis shows that $A/M \simeq \ln M$. This last applies for $\beta \neq 0$ also.

(g) *An heuristic approach*

We may now return to the general problem of solving equations (16)–(18). Clearly, the first step in any particular situation is to calculate B, M^{-1} and Gn. It may be remarked in passing that M^{-1} can also be regarded as a Graetz number, for it has been shown [4] that, for such a situation

$$Gz = \rho_f C_f W_o \delta_o^2 / kL = M^{-1}$$

using (15) and (19). If any of B, M^{-1} or $Gn \ll 1$, then one of the solutions above may be employed. In doing this, we are assuming that the basically parabolic equations (16) and (18) will have solutions that rapidly settle down to their unique similarity form. In practice (for plastics extruders) we find that B is significant, Gn is relatively small and that β and M are of order unity.

We shall try to make use of the fact that we have available well-tried numerical methods for solving the equations with $M = 0$. We shall therefore attempt what is essentially an expansion in powers of M^{-1}. We find it convenient to use (1) in the integrated form

$$v(\delta) = -\frac{\partial}{\partial z}\int_0^\delta w\,dy = -\frac{\partial Q}{\partial z}. \tag{57}$$

Neglecting convection terms in (3), w and θ are given by

$$\frac{\partial}{\partial y}\left\{e^{-b\theta^{(0)}}\left(\frac{\partial w^{(0)}}{\partial y}\right)^{1-2\beta}\right\} = 0 \tag{58}$$

and

$$k\frac{\partial^2\theta^{(0)}}{\partial y^2} + C_o e^{-b\theta^{(0)}}\left(\frac{\partial w^{(0)}}{\partial y}\right)^{2-2\beta} = 0 \tag{59}$$

with

$$w^{(0)}(0) = W_o,\ \theta^{(0)}(0) = \theta_0,\ w^{(0)}(\delta) = 0,\ \theta^{(0)}(\delta) = 0. \tag{60}$$

If δ is given the solution is unique. But since the boundary condition (7) gives

$$v(\delta) = \frac{k}{\rho_f \Lambda}\frac{\partial\theta}{\partial y}(\delta) \tag{61}$$

it is clear from (57) that it is information about Q that will be carried forward to larger values of z. Hence we suppose that $Q(z)$ is given and thus (58)–(60) are taken to imply $\delta(z)$. Equation (61) then yields $\partial Q/\partial z$. If we now try to improve our approximation for θ in (61) we are tempted to solve for $\theta = \theta^{(0)} + \theta^{(1)}$, $w = w^{(0)} + w^{(1)}$, where we have

$$\frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial z} = 0 \tag{62}$$

$\partial w^{(0)}/\partial z$ being supposed known,

$$\frac{\partial}{\partial y}\left(e^{-b(\theta^{(0)}+\theta^{(1)})}\frac{\partial(w^{(0)}+w^{(1)})^{1-2\beta}}{\partial y}\right) = 0 \tag{63}$$

$$k\frac{\partial^2\theta^{(0)}+\theta^{(1)}}{\partial y^2} + C_o e^{-b(\theta^{(0)}+\theta^{(1)})}\frac{\partial(w^{(0)}+w^{(1)})^{2-2\beta}}{\partial y} = \rho_f C_f\left(w^{(0)}\frac{\partial\theta^{(0)}}{\partial z} + v^{(0)}\frac{\partial\theta^{(0)}}{\partial y}\right) \tag{64}$$

with

$$w^{(1)}(0) = \theta^{(1)}(0) = w^{(1)}(\delta) = \theta^{(1)}(\delta) = 0 \tag{65}$$

and

$$v^{(1)}(\delta) = \frac{k}{\rho_f \Lambda}\frac{\partial\theta^{(1)}}{\partial y}(\delta). \tag{66}$$

This would clearly be a laborious process to undertake in every case, and so an heuristic alternative has been used that assumes $\partial\theta^{(1)}/\partial y$ to bear a fixed relation to $\partial\theta^{(0)}/\partial y$ at $y = \delta$. This is then interpreted in terms of an apparent Λ^* chosen so that

$$\frac{\partial\theta^{(0)}+\theta^{(1)}}{\partial y}\cdot\Lambda^* = \Lambda\frac{\partial\theta^{(0)}}{\partial y}. \tag{67}$$

The ratio Λ^*/Λ can be derived formally from the similarity solutions given in subsections 3(a), 3(b) and 3(c) in the form

$$\Lambda^* = \Lambda(1 + CM^{-1}) \tag{68}$$

where $0 < C < 1$ and C is related to $\hat{\theta}$. Some details are given in the Appendix of [2], where it becomes clear that the value $C = \hat{\theta}$ taken by Shapiro [5] is too high. This value was obtained by supposing that all the heat required to raise the temperature of the melting fluid to the mean temperature $\hat{\theta}$ came from conduction at the melting interface. If however one assumes that the heat required to raise the temperature of the fluid melted at the interface $y = \delta$ to $\hat{\theta}$ comes equally from the interfaces $y = 0$ and $y = \delta$, then

$$C = \tfrac{1}{2}\hat{\theta}. \tag{69}$$

An exact expansion shows that in the case $B = 0$, $Gn = 0$,

$$C = \tfrac{5}{16}\hat{\theta}.$$

By dealing almost entirely with similarity solutions we have avoided serious consideration of any initial conditions at $z = 0$. If we are given $Q(0)$ we can estimate $\delta(0)$ and hence obtain $\alpha(0)$. Using a particular similarity solution yielding a given A, this means that

$$\zeta(0) = \frac{1}{2A}\alpha^2(0) \quad \text{and not zero}$$

and so

$$\alpha(z) = \left(\frac{2Az}{L} + \alpha^2(0)\right)^{\frac{1}{2}}.$$

In practice $\alpha^2(0)$ soon becomes negligible.

A preliminary application of this work to the melting of polymer granules is given in another report [6]; further work is in progress.

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SUR LA FUSION DES SOLIDES AU VOISINAGE D'UN INTERFACE CHAUD MOBILE, AVEC REFERENCE PARTICULIERE AU CAS DES LITS DE POLYMERES GRANULES

Résumé—On présente quelques solutions de similitude pour les profils de vitesse et de température à l'intérieur du film mince de fluide qui se forme entre un interface chaud en mouvement et le solide en fusion. L'utilisation de variables adimensionnelles met en évidence le rôle des divers groupements sans dimension, et en particulier lorsqu'on considère le cisaillement d'un fluide dont la viscosité dépend de la température.

Une méthode de solution euristique simplifiée est discutée pour le problème complet dans le cas de l'extrusion avec déformation plastique.

DAS SCHMELZEN VON FESTKÖRPERN NAHE EINER HEISSEN BEWEGTEN SCHICHT, INSBESONDERE VON POLYMER-GRANULAT-BETTEN

Zusammenfassung—Ähnlichkeitslösungen werden angegeben für Geschwindigkeits-und Temperaturprofile im dünnen Film eines Fluids, der sich zwischen der bewegten heissen Zwischenschicht und dem schmelzenden Festkörper bildet. Die Benützung von dimensionslosen Variablen zeigt die Rolle der verschiedenen dimensionslosen Gruppen, insbesondere für schubspannungs- und temperaturabhängige viskose Fluide. Eine vereinfachte heuristische Lösungsmethode des Gesamtproblems für Plastik-Extruder wird diskutiert.

О ПЛАВЛЕНИИ ТВЕРДЫХ ТЕЛ ВБЛИЗИ ДВИЖУЩЕЙСЯ ГОРЯЧЕЙ ПОВЕРХНОСТИ РАЗДЕЛА ПРИМЕНИТЕЛЬНО К СЛОЯМ ГРАНУЛИРОВАННЫХ ПОЛИМЕРОВ

Аннотация — Приводятся некоторые решения подобия для профилей скорости и температуры в тонкой жидкой пленке, образующейся между движущейся горячей поверхностью раздела и плавящимся твердым телом. С помощью безразмерных переменных выявляется значение различных безразмерных групп, в частности при рассмотрении жидкости, вязкость которой зависит от напряжения сдвига и температуры.

Анализируется упрощенный эвристический метод решения задачи применительно к случаю пластицирующих экструдеров.